

Hybrid stress quadrilateral finite element approximation for stochastic plane elasticity equations ^{*}

Xiaojing Xu[†], Wenwen Fan[‡], Xiaoping Xie [§]

School of Mathematics, Sichuan University, Chengdu 610064, China

Abstract

This paper considers stochastic hybrid stress quadrilateral finite element analysis of plane elasticity equations with stochastic Young's modulus and stochastic loads. Firstly, we apply Karhunen-Loève expansion to stochastic Young's modulus and stochastic loads so as to turn the original problem into a system containing a finite number of deterministic parameters. Then we deal with the stochastic field and the space field by k -version/ p -version finite element methods and a hybrid stress quadrilateral finite element method, respectively. We show that the derived a priori error estimates are uniform with respect to the Lamé constant $\lambda \in (0, +\infty)$. Finally, we provide some numerical results.

Keywords. stochastic plane elasticity Karhunen-Loève expansion hybrid stress finite element $k \times h$ -version $p \times h$ -version uniform error estimate

1 Introduction

Let $D \subset R^2$ be a bounded, connected, convex and open set with boundary $\partial D = \partial D_0 \cup \partial D_1$ and $\text{meas}(\partial D_0) > 0$, and let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where Ω , \mathcal{F} , \mathcal{P} denote respectively the set of outcomes, the σ -algebra of subsets of Ω and the probability measure. Consider the following stochastic plane elasticity equations: for

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[†]Email: xuxiaojing0603@126.com

[‡]Email: fwwen123@126.com

[§]Corresponding author. Email: xpxie@scu.edu.cn

almost everywhere (a.e.) $\theta \in \Omega$

$$\begin{cases} -\mathbf{div}\boldsymbol{\sigma}(\cdot, \theta) = \mathbf{f}(\cdot, \theta), & \text{in } D, \\ \boldsymbol{\sigma}(\cdot, \theta) = \mathcal{C}\epsilon(\mathbf{u}(\cdot, \theta)), & \text{in } D, \\ \mathbf{u}(\cdot, \theta)|_{\partial D_0} = 0, \boldsymbol{\sigma}(\cdot, \theta)\mathbf{n}|_{\partial D_1} = \mathbf{g}(\cdot, \theta), \end{cases} \quad (1.1)$$

where $\boldsymbol{\sigma} : \overline{D} \times \Omega \rightarrow R_{sym}^{2 \times 2}$ denotes the symmetric stress tensor field, $\mathbf{u} : \overline{D} \times \Omega \rightarrow R^2$ the displacement field, $\epsilon(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$ the strain with $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$ for $\mathbf{x} = (x_1, x_2)$, $\mathbf{f} : D \times \Omega \rightarrow R^2$ the body loading density and $\mathbf{g} : \partial D_1 \times \Omega \rightarrow R^2$ the surface traction, \mathbf{n} the unit outward vector normal to ∂D , \mathcal{C} the elasticity modulus tensor with

$$\mathcal{C}\epsilon(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda\text{div}\mathbf{u}\mathbf{I},$$

\mathbf{I} the 2×2 identity tensor, and μ, λ the Lamé parameters given by $\mu = \frac{\tilde{E}}{2(1+\nu)}$, $\lambda = \frac{\tilde{E}\nu}{(1+\nu)(1-2\nu)}$ for plane strain problems and by $\mu = \frac{\tilde{E}}{2(1+\nu)}$, $\lambda = \frac{\tilde{E}}{(1+\nu)(1-\nu)}$ for plane stress problems, with $\nu \in (0, 0.5)$ the Poisson ratio and $\tilde{E} : D \times \Omega \rightarrow R$ the Young's modulus which is stochastic with

$$0 < e_{min} \leq \tilde{E}(\mathbf{x}, \theta) \leq e_{max} \quad \text{a.e. in } D \times \Omega \quad (1.2)$$

for positive constants e_{min} and e_{max} . Since in the analysis of this paper we need to use an explicit form of \tilde{E} , we rewrite the second equation of (1.1) as

$$\boldsymbol{\sigma}(\cdot, \theta) = \tilde{E}\mathbf{C}\epsilon(\mathbf{u}(\cdot, \theta)), \quad (1.3)$$

where the tensor $\mathbf{C} := \frac{1}{E}\mathcal{C}$ depends only on the Poisson ratio ν .

It is well-known that the standard 4-node displacement quadrilateral element (abbr. bilinear element) yields poor results for deterministic plane elasticity equations with bending and, for deterministic plane strain problems, at the nearly incompressible limit. To improve its performance, Wilson et al. [26, 24] developed methods of incompatible modes by enriching the standard (compatible) displacement modes with internal incompatible displacements. Pian and Sumihara [17] proposed a hybrid stress quadrilateral element (PS element) based on Hellinger-Reissner variational principle, where the displacement vector is approximated by isoparametric bilinear interpolations, and the stress tensor by a piecewise-independent 5-parameter mode. Xie and Zhou [31, 32] derived robust 4-node hybrid stress quadrilateral elements by optimizing stress modes with a so-called energy-compatibility condition, i.e. the assumed stress terms are orthogonal to the enhanced

strains caused by Wilson bubble displacements. In [35] Zhou and Xie gave a unified analysis for some hybrid stress/strain quadrilateral methods, but the upper bound in the error estimate is not uniform with respect to the Lamé parameter λ . Yu, Xie and Carstensen [33] derived uniform convergence results for the hybrid stress methods in [17] and [31], in the sense that the error bound is independent of λ .

In the numerical analysis of stochastic partial differential equations, stochastic finite element methods, which employ finite elements in the space domain, have gained much attention in the past two decades. In the probability domain, the stochastic finite element methods use two types of approximation methods, statistical approximation and non-statistical approximation. Monte Carlo sampling (MCs) is one of the most commonly used statistical approximation methods [22]. In MCs, one generates realizations of stochastic terms so as to make the problem deterministic, and only needs to compute the deterministic problem repeatedly, and collect an ensemble of solutions, through which statistical information, such as mean and variance, can be obtained. The disadvantage of MCs lies in the need of a large amount of calculations and its low convergence rate. There are also some variants of MCs such as quasi Monte Carlo [6] and the stochastic collocation method [2, 14, 15, 16].

Non-statistical approximation methods mainly contain perturbation methods, Neumann series expansion methods [10] and so on at the beginning. But these methods are limited to the magnitude of uncertainties of stochastic terms and the accuracy of calculation. Later, polynomial approximation is used for the stochastic part. For example, Polynomial chaos (PC) expansion is applied in [27, 10] to represent solutions formally and obtain solutions by solving the expansion coefficients [9, 13]. Generalized polynomial chaos (gPC) is used to express solutions in [12, 28, 29]. According to [30], one can achieve exponential convergence when optimum gPC is chosen. Subsequently, it was further generalized [1, 7] that p version, k version and p-k-version finite element methods could be used for the approximation of the stochastic part.

So far, there are very limited studies on the numerical solution of the stochastic plane elasticity equations (1.1). In [11] a generalized n th order stochastic perturbation technique is implemented in conjunction with linear finite elements to model a 1D linear elastostatic problem with a single random variable. In [9] the numerical solution of problem (1.1) is considered with stochastic Young's modulus \tilde{E} , where PC approximation and bilinear

finite elements are applied respectively to the stochastic domain and the space domain. We refer to [5, 25] for some other related studies. In this contribution, we shall propose and analyze stochastic $k \times h$ -version and $p \times h$ -version finite element methods for the problem (1.1), where we use k -version/ p -version finite element methods for the stochastic domain and PS hybrid stress quadrilateral finite element for the space domain.

We arrange the paper as follows. In Section 2 we show stochastic mixed variational formulations of (1.1), and give the existence and uniqueness of the weak solution. Section 3 discusses the approximation of the stochastic coefficient and stochastic loads, as well as the truncated stochastic mixed variational formulations. Section 4 analyzes the proposed stochastic $k \times h$ -version and $p \times h$ -version finite element methods and derives uniform a priori error estimates. Finally, Section 5 provides some numerical results.

2 Stochastic mixed variational formulations

2.1 Notations

For the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and an integer m , denote

$$L_P^m(\Omega) := \left\{ Y \mid Y \text{ is a random variable in } (\Omega, \mathcal{F}, \mathcal{P}) \text{ with } \int_{\Omega} |Y(\theta)|^m dP(\theta) < +\infty \right\}.$$

If $Y \in L_P^1(\Omega)$, we denote its expected value by

$$E[Y] = \int_{\Omega} Y(\theta) dP(\theta) = \int_R y dF(y), \quad (2.1)$$

where F is the distribution probability measure of Y , given by $F(B) = P(Y^{-1}(B))$ for any borel set B in R . Assume that $F(B)$ is absolutely continuous with respect to Lebesgue measure, then there exists a density function for Y , $\rho : R \rightarrow [0, +\infty)$, such that

$$E[Y] = \int_R y \rho(y) dy. \quad (2.2)$$

We denote by $H^m(D)$ the usual Sobolev space consisting of functions defined on the domain D , with all derivatives of order up to m square-integrable. Let $(\cdot, \cdot)_{H^m(D)}$ be the usual inner product on $H^m(D)$. The norm $\|\cdot\|_m$ on $H^m(D)$ deduced by $(\cdot, \cdot)_{H^m(D)}$ is given by

$$\|v\|_m := \left(\sum_{0 \leq j \leq m} |v|_j^2 \right)^{1/2} \text{ with the semi-norm } |v|_j := \left(\sum_{|\alpha|=j} \|D^\alpha v\|_0^2 \right)^{1/2}.$$

In particular, $L^2(D) := H^0(D)$. Denote

$$L^\infty(D) := \{w : \|w\|_\infty := \text{esssup}_{x \in D} |w(x)| < \infty\}.$$

We define the following stochastic Sobolev spaces:

$$L_P^2(\Omega; H^m(D)) := \{w : w \text{ is strongly measurable with } w(\cdot, \theta) \in H^m(D) \text{ for } \theta \in \Omega \text{ and } \|w\|_{\tilde{m}} < +\infty\},$$

$$L_P^\infty(\Omega; L^\infty(D)) := \{w : w \text{ is strongly measurable with } w(\cdot, \theta) \in L^\infty(D) \text{ for } \theta \in \Omega \text{ and } \|w\|_\infty < +\infty\},$$

where the norms $\|\cdot\|_{\tilde{m}}$, $\|\cdot\|_\infty$ are respectively defined as

$$\|w\|_{\tilde{m}} := (E[\|w(\cdot, \theta)\|_m^2])^{\frac{1}{2}}, \quad \|w\|_\infty := \text{esssup}_{\theta \in \Omega} \|w(\cdot, \theta)\|_\infty. \quad (2.3)$$

On the other hand, since stochastic functions intrinsically have different structures with respect to $\theta \in \Omega$ and $\mathbf{x} \in D$, we follow [1] to introduce tensor spaces for the analysis of numerical approximation. Let $X_1(\Omega)$, $X_2(D)$ be Hilbert spaces. The tensor spaces $X_1(\Omega) \otimes X_2(D)$ is the completion of formal sums $\phi(\theta, \mathbf{x}) = \sum_{i=1, \dots, n} u_i(\theta) v_i(\mathbf{x})$, $u_i \in X_1(\Omega)$, $v_i \in X_2(D)$, with respect to the inner product $(\phi, \hat{\phi})_{X_1 \otimes X_2} := \sum_{i,j} (u_i, \hat{u}_j)_{X_1} (v_i, \hat{v}_j)_{X_2}$. Then, for the tensor space $L_P^2(\Omega) \otimes H^m(D)$, we have the following isomorphism:

$$L_P^2(\Omega; H^m(D)) \simeq L_P^2(\Omega) \otimes H^m(D).$$

For convenience, we use the notation $a \lesssim b$ to represent that there exists a generic positive constant C such that $a \leq Cb$, where C is independent of the Lamé constant λ and the mesh parameters h , k , the polynomial degree p in the stochastic $k \times h$ -version and $p \times h$ -version finite element methods.

2.2 Weak formulations

Introduce the spaces

$$V_D := \{v \in H^1(D)^2 : v|_{\partial D_0} = 0\},$$

$$\Sigma_D := \begin{cases} L^2(D; R_{sym}^{2 \times 2}) := \{\tau : D \rightarrow R^{2 \times 2} \mid \tau_{ij} \in L^2(D), \tau_{ij} = \tau_{ji}, i, j = 1, 2\}, & \text{if } \text{meas}(\partial D_1) > 0, \\ \{\tau \in L^2(D; R_{sym}^{2 \times 2}) : \int_D \text{tr} \tau \, d\mathbf{x} = 0 \text{ with trace } \text{tr} \tau := \tau_{11} + \tau_{22}\}, & \text{if } \partial D_1 = \emptyset. \end{cases}$$

Then the weak problem for the model (1.1) reads as: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D)$ such that

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \mathbf{u}) = 0, & \forall \boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D), \\ b(\boldsymbol{\sigma}, \mathbf{v}) = \ell(\mathbf{v}), & \forall \mathbf{v} \in L_P^2(\Omega; V_D), \end{cases} \quad (2.4)$$

where the bilinear forms $a(\cdot, \cdot) : L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; \Sigma_D) \rightarrow R$, $b(\cdot, \cdot) : L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D) \rightarrow R$ and the linear form $\ell : L_P^2(\Omega; V_D) \rightarrow R$ are defined respectively by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := E\left[\int_D \frac{1}{\widetilde{E}} \boldsymbol{\sigma} : \mathbf{C}^{-1} \boldsymbol{\tau} d\mathbf{x}\right] = \int_\Omega \int_D \frac{1}{\widetilde{E}} \boldsymbol{\sigma} : \mathbf{C}^{-1} \boldsymbol{\tau} d\mathbf{x} dP(\theta), \quad (2.5)$$

$$b(\boldsymbol{\tau}, \mathbf{u}) := E\left[\int_D \boldsymbol{\tau} : \boldsymbol{\epsilon}(\mathbf{u}) d\mathbf{x}\right] = \int_\Omega \int_D \boldsymbol{\tau} : \boldsymbol{\epsilon}(\mathbf{u}) d\mathbf{x} dP(\theta), \quad (2.6)$$

$$\ell(\mathbf{v}) := E\left[\int_D \mathbf{f} \mathbf{v} d\mathbf{x} + \int_{\partial D_1} \mathbf{g} \cdot \mathbf{v} ds\right] = \int_\Omega \int_D \mathbf{f} \mathbf{v} d\mathbf{x} dP(\theta) + \int_\Omega \int_{\partial D_1} \mathbf{g} \cdot \mathbf{v} ds dP(\theta). \quad (2.7)$$

Here $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^2 \sigma_{ij} \tau_{ij}$.

It is easy to see that the following continuity conditions hold: for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D)$, $\mathbf{v} \in L_P^2(\Omega; V_D)$,

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \lesssim \|\boldsymbol{\sigma}\|_{\widetilde{0}} \|\boldsymbol{\tau}\|_{\widetilde{0}}, \quad b(\boldsymbol{\tau}, \mathbf{v}) \lesssim \|\boldsymbol{\tau}\|_{\widetilde{0}} |\mathbf{v}|_{\widetilde{1}}, \quad \ell(\mathbf{v}) \lesssim (\|\mathbf{f}\|_{\widetilde{0}} + \|\mathbf{g}\|_{\widetilde{0}, \partial D_1}) |\mathbf{v}|_{\widetilde{1}}. \quad (2.8)$$

According to the theory of mixed finite element methods [3][4], we need the following two stability conditions for the well-posedness of the weak problem (2.4):

(A) Kernel-coercivity: for any $\boldsymbol{\tau} \in Z^0 := \{\boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D) : b(\boldsymbol{\tau}, \mathbf{v}) = 0, \forall \mathbf{v} \in L_P^2(\Omega; V_D)\}$ it holds

$$\|\boldsymbol{\tau}\|_{\widetilde{0}}^2 \lesssim a(\boldsymbol{\tau}, \boldsymbol{\tau}). \quad (2.9)$$

(B) Inf-sup condition: for any $\mathbf{v} \in L_P^2(\Omega; V_D)$ it holds

$$|\mathbf{v}|_{\widetilde{1}} \lesssim \sup_{0 \neq \boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D)} \frac{b(\boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{\widetilde{0}}}. \quad (2.10)$$

Theorem 2.1. *The uniform stability conditions (A) and (B) hold.*

Proof. For any $\boldsymbol{\tau} \in Z^0$, we have, a.e. $\theta \in \Omega$, $\boldsymbol{\tau}(\cdot, \theta) \in \{\boldsymbol{\tau} \in \Sigma_D : \int_D \boldsymbol{\tau} : \boldsymbol{\epsilon}(\mathbf{v}) d\mathbf{x} = 0 \quad \forall \mathbf{v} \in V_D\}$. According to Theorem 2.1 in [33] and the assumption (1.2), it holds

$$\int_D \boldsymbol{\tau}(\cdot, \theta) : \boldsymbol{\tau}(\cdot, \theta) d\mathbf{x} \lesssim \int_D \frac{1}{\widetilde{E}} \boldsymbol{\tau}(\cdot, \theta) : \mathbf{C}^{-1}(\cdot, \theta) \boldsymbol{\tau}(\cdot, \theta) d\mathbf{x},$$

which leads to

$$\int_\Omega \int_D \boldsymbol{\tau} : \boldsymbol{\tau} d\mathbf{x} dP(\theta) \lesssim \int_\Omega \int_D \frac{1}{\widetilde{E}} \boldsymbol{\tau} : \mathbf{C}^{-1} \boldsymbol{\tau} d\mathbf{x} dP(\theta),$$

i.e. (A) holds.

Let $\mathbf{v} \in L_P^2(\Omega; V_D)$ and notice $\boldsymbol{\epsilon}(\mathbf{v}) \in L_P^2(\Omega; \Sigma_D)$. Then

$$|\boldsymbol{\epsilon}(\mathbf{v})|_{\widetilde{0}} \leq \sup_{\boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D) \setminus \{0\}} \frac{\int_\Omega \int_D \boldsymbol{\tau} : \boldsymbol{\epsilon}(\mathbf{v}) d\mathbf{x} dP(\theta)}{\|\boldsymbol{\tau}\|_{\widetilde{0}}}.$$

Hence (B) follows from the equivalence between the two norms $|\boldsymbol{\epsilon}(\mathbf{v})|_{\widetilde{0}}$ and $|\mathbf{v}|_{\widetilde{1}}$ on $L_P^2(\Omega; V_D)$. \square

In view of the above conditions, we immediately obtain the following well-posedness result:

Theorem 2.2. *Assume that $\mathbf{f} \in L_P^2(\Omega, L^2(D)^2)$, $\mathbf{g} \in L_P^2(\Omega, L^2(\partial D_1)^2)$. Then the weak problem (2.4) admits a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D)$ such that*

$$\|\boldsymbol{\sigma}\|_{\tilde{0}} + \|\mathbf{u}\|_{\tilde{1}} \lesssim \|\mathbf{f}\|_{\tilde{0}} + \|\mathbf{g}\|_{\tilde{0}, \partial D_1}. \quad (2.11)$$

3 Truncated stochastic mixed variational formulations

In order to solve the weak problem (2.4) by deterministic numerical methods, we firstly approximate the stochastic coefficient \tilde{E} and the loads \mathbf{f} , \mathbf{g} by using a finite number of random variables; we refer to [21] for several approximation approaches. Here, we only consider the Karhunen-Loève(K-L) expansion.

3.1 Karhunen-Loève(K-L) expansion

For any stochastic process $\phi(\mathbf{x}, \theta) \in L_P^2(\Omega; L^2(D))$ with covariance function $\text{cov}[\phi](\mathbf{x}_1, \mathbf{x}_2) : D \times D \rightarrow R$, which is bounded, symmetric and positive definitely. Let $\{(\lambda_n, b_n)\}_{n=1}^{\infty}$ be the sequence of eigenpairs satisfying

$$\int_D \text{cov}[\phi](\mathbf{x}_1, \mathbf{x}_2) b_n(\mathbf{x}_2) d\mathbf{x}_2 = \lambda_n b_n(\mathbf{x}_1), \quad (3.1)$$

$$\sum_{n=1}^{+\infty} \lambda_n = \int_D \text{cov}[\phi](\mathbf{x}, \mathbf{x}) d\mathbf{x}, \quad \int_D b_i(\mathbf{x}) b_j(\mathbf{x}) d\mathbf{x} = \delta_{ij}, \quad i, j = 1, 2, \dots, \quad (3.2)$$

and $\lambda_1 \geq \lambda_2 \geq \dots > 0$. Then the Karhunen-Loève(K-L) expansion of $\phi(\mathbf{x}, \theta)$ is given by

$$\phi(\mathbf{x}, \theta) = E[\phi](\mathbf{x}) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n(\mathbf{x}) Y_n(\theta), \quad (3.3)$$

and the truncated K-L expansion of $\phi(\mathbf{x}, \theta)$ is

$$\phi_N(\mathbf{x}, \theta) = E[\phi](\mathbf{x}) + \sum_{n=1}^N \sqrt{\lambda_n} b_n(\mathbf{x}) Y_n(\theta). \quad (3.4)$$

Here $\{Y_n\}_{n=1}^{\infty}$ are mutually uncorrelated with mean zeros and unit variance with $Y_n(\theta) = \frac{1}{\sqrt{\lambda_n}} \int_D (\phi(\mathbf{x}, \theta) - E[\phi](\mathbf{x})) b_n(\mathbf{x}) d\mathbf{x}$.

By Mercer's theorem [20], it holds

$$\sup_{\mathbf{x} \in D} E[(\phi - \phi_N)^2](\mathbf{x}) = \sup_{\mathbf{x} \in D} \sum_{n=N+1}^{+\infty} \lambda_n b_n^2(\mathbf{x}) \rightarrow 0. \quad \text{as } N \rightarrow \infty. \quad (3.5)$$

In what follows we show the estimation of the truncated error $\phi - \phi_N$ in norms $\|\cdot\|_{\tilde{0}}$ and $\|\cdot\|_{\infty}$, respectively.

From (3.2) it follows

$$\|\phi - \phi_N\|_{\tilde{0}}^2 = \sum_{n=N+1}^{+\infty} \lambda_n \quad \text{and} \quad \|\phi - \phi_N\|_{\tilde{0}} \rightarrow 0 \quad \text{as } N \rightarrow +\infty. \quad (3.6)$$

Obviously the convergence rate of $\|\phi - \phi_N\|_{\tilde{0}}$ is strongly depending on the decay rate of the eigenvalues λ_n , which ultimately depends on the regularity of the covariance function $\text{cov}[\phi]$. Generally, the smoother the covariance is, the faster the eigenvalues decay, which implies the faster $\|\phi - \phi_N\|_{\tilde{0}}$ converges to zero. Now we quote from [23] the following definition (Definition 3.1, which are related to the regularity of $\text{cov}[\phi]$) and lemma (Lemma 3.1, which gives the decay rate of the eigenvalues λ_n).

Definition 3.1. [23] *The covariance function $\text{cov}[\phi] : D \times D \rightarrow R$ is said to be piecewise analytic/smooth on $D \times D$ if there exists a finite family $(D_j)_{1 \leq j \leq J} \subset R^2$ of open hypercubes such that $\overline{D} \subseteq \cup_{j=1}^J \overline{D}_j$, $D_j \cap D_{j'} = \emptyset$, $\forall j \neq j'$ and $\text{cov}[\phi]|_{D_j \times D_{j'}}$ has an analytic/smooth continuation in a neighbourhood of $\overline{D}_j \times \overline{D}_{j'}$ for any pair (j, j') .*

Lemma 3.1. [23] *If $\text{cov}[\phi]$ is piecewise analytic on $D \times D$, then for the eigenvalue sequence $\{\lambda_n\}_{n \geq 1}$, there exist constants c_1, c_2 depending only on $\text{cov}[\phi]$ such that*

$$0 \leq \lambda_n \leq c_1 e^{-c_2 n^{1/2}}, \quad \forall n \geq 1. \quad (3.7)$$

If $\text{cov}[\phi]$ is piecewise smooth on $D \times D$, then for any constant $s > 0$ there exists a constant c_s depending only on $\text{cov}[\phi]$ and s , such that

$$0 \leq \lambda_n \leq c_s n^{-s}, \quad \forall n \geq 1. \quad (3.8)$$

By Lemma 3.1, we immediately have the following convergence results.

Lemma 3.2. *If $\text{cov}[\phi]$ is piecewise analytic on $D \times D$, then there exists constants c_1, c_2 depending only on $\text{cov}[\phi]$ such that*

$$\|\phi - \phi_N\|_{\tilde{0}} \leq \frac{2c_1}{c_2^2} (1 + c_2 N^{1/2}) e^{-c_2 N^{1/2}}, \quad \forall N \geq 1. \quad (3.9)$$

If $\text{cov}[\phi]$ is piecewise smooth on $D \times D$, then for any $s > 0$ there exists C_s depending only on $\text{cov}[\phi]$ and s , such that

$$\|\phi - \phi_N\|_{\tilde{0}} \leq C_s N^{-s}, \quad \forall N \geq 1. \quad (3.10)$$

To estimate $\|\phi - \phi_N\|_\infty$, we make the following assumption:

Assumption 3.1. *The random variables $\{Y_n(\theta)\}_{n=1}^\infty$ in the K-L expansion are independent and uniformly bounded with*

$$\|Y_n(\theta)\|_{L^\infty(\Omega)} \leq C_Y, \quad \forall n \geq 1,$$

where C_Y is a positive constant.

Lemma 3.3. *[8, 23] Suppose Assumption 3.1 holds. If $\text{cov}[\phi]$ is piecewise analytic on $D \times D$, then there exist a constant $c > 0$ such that, for any $s > 0$, it holds*

$$\|\phi - \phi_N\|_\infty \leq C e^{-c(1/2-s)N^{1/2}}, \quad \forall N \geq 1, \quad (3.11)$$

where C is a positive constant depending on $s, c, \text{cov}[\phi]$ and J given in Definition 3.1. If $\text{cov}[\phi]$ is piecewise smooth on $D \times D$, then for any $t > 0, r > 0$, it holds

$$\|\phi - \phi_N\|_\infty \leq C' N^{1-t(1-r)/2}, \quad \forall N \geq 1, \quad (3.12)$$

where C' is a positive constant depending on $t, r, \text{cov}[\phi]$ and J .

Remark 3.1. *We note that we need to solve the integral equation (3.1) to obtain the K-L expansion (3.3). For some special covariance functions, the equation can be solved analytically [10], but for more general cases numerical methods are required [8, 18, 23].*

3.2 Finite dimensional approximations of \tilde{E} , \mathbf{f} , \mathbf{g}

In this section, we use the K-L expansion to approximate \tilde{E} , \mathbf{f} and \mathbf{g} .

For \tilde{E} , assume its truncated K-L expansion is of the form

$$\tilde{E}_N(\mathbf{x}, \theta) = \tilde{E}_N(\mathbf{x}, Y_1(\theta), \dots, Y_N(\theta)) = E[\tilde{E}](\mathbf{x}) + \sum_{n=1}^N \sqrt{\tilde{\lambda}_n} \tilde{b}_n(\mathbf{x}) Y_n(\theta), \quad (3.13)$$

where $\{(\tilde{\lambda}_n, \tilde{b}_n(\mathbf{x}))\}_{n=1}^N$ and $\{Y_n(\theta)\}_{n=1}^N$ are the corresponding eigenpairs and random variables, respectively.

As for $\mathbf{f} = (f_1, f_2)^T$ and $\mathbf{g} = (g_1, g_2)^T$, we need to apply the K-L expansion to each of their components. In this paper, following similar ways as in [1, 2] to avoid use of more notations, we assume the truncated K-L expansions of \mathbf{f} and \mathbf{g} take the following forms:

$$\mathbf{f}_N(\mathbf{x}, \theta) = \mathbf{f}_N(\mathbf{x}, Y_1(\theta), \dots, Y_N(\theta)) = \begin{pmatrix} f_{1N} \\ f_{2N} \end{pmatrix} = \begin{pmatrix} E[f_1](\mathbf{x}) \\ E[f_2](\mathbf{x}) \end{pmatrix} + \sum_{n=1}^N \begin{pmatrix} \sqrt{\hat{\lambda}_{1n}} \hat{b}_{1n}(\mathbf{x}) \\ \sqrt{\hat{\lambda}_{2n}} \hat{b}_{2n}(\mathbf{x}) \end{pmatrix} Y_n(\theta), \quad (3.14)$$

$$\mathbf{g}_N(\mathbf{x}, \theta) = \mathbf{g}_N(\mathbf{x}, Y_1(\theta), \dots, Y_N(\theta)) = \begin{pmatrix} g_{1N} \\ g_{2N} \end{pmatrix} = \begin{pmatrix} E[g_1](\mathbf{x}) \\ E[g_2](\mathbf{x}) \end{pmatrix} + \sum_{n=1}^N \begin{pmatrix} \sqrt{\bar{\lambda}_{1n}} \bar{b}_{1n}(\mathbf{x}) \\ \sqrt{\bar{\lambda}_{2n}} \bar{b}_{2n}(\mathbf{x}) \end{pmatrix} Y_n(\theta), \quad (3.15)$$

where $\{(\hat{\lambda}_{in}, \hat{b}_{in}(\mathbf{x}))\}_{n=1}^N$, $\{(\bar{\lambda}_{in}, \bar{b}_{in}(\mathbf{x}))\}_{n=1}^N, i = 1, 2$ are the corresponding eigenpairs.

Remark 3.2. In practice, the Young's modulus \tilde{E} , the body force \mathbf{f} and the surface load \mathbf{g} may be independent. In such cases, the random variables $\{Y_n(\theta)\}_{n=1}^N$ in the truncated K-L expansions (3.13)-(3.15) for $\tilde{E}, f_1, f_2, g_1, g_2$ may be different from each other. However, the analysis of this paper still applies to these cases.

3.3 Truncated mixed formulations

By replacing $\tilde{E}, \mathbf{f}, \mathbf{g}$ with their truncated forms $\tilde{E}_N, \mathbf{f}_N, \mathbf{g}_N$ in the bilinear form $a(\cdot, \cdot)$, given in (2.5), and the linear form $\ell(\cdot)$, given in (2.7), we can obtain the following modified mixed variational formulations for the weak problem (2.4): find $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D)$ such that

$$\begin{cases} a_N(\boldsymbol{\sigma}_N, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \mathbf{u}_N) = 0, & \forall \boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D), \\ b(\boldsymbol{\sigma}_N, \mathbf{v}) = \ell_N(\mathbf{v}), & \forall \mathbf{v} \in L_P^2(\Omega; V_D). \end{cases} \quad (3.16)$$

We recall that $\{Y_n(\theta)\}_{n=1}^N$ are the random variables used in the K-L expansions of \tilde{E} , \mathbf{f} and \mathbf{g} , which are assumed to satisfy Assumption 3.1. In what follows we denote

$$Y := (Y_1, Y_2, \dots, Y_N), \quad \Gamma_n := Y_n(\Omega) \subset R, \quad \Gamma := \prod_{n=1}^N \Gamma_n, \quad (3.17)$$

and let $\rho : \Gamma \rightarrow R$ be the joint probability density function of random vector Y with $\rho \in L^\infty(\Gamma)$. According to Doob-Dynkin lemma [19], the weak solution of the modified problem (3.16) can be described by the random vector Y as

$$\mathbf{u}_N(\mathbf{x}, \theta) = \mathbf{u}_N(\mathbf{x}, Y), \quad \boldsymbol{\sigma}_N(\mathbf{x}, \theta) = \boldsymbol{\sigma}_N(\mathbf{x}, Y),$$

and, by denoting $\mathbf{y} := (y_1, y_2, \dots, y_N)$, the corresponding strong formulation for (3.16) is of the form

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma}_N(\mathbf{x}, \mathbf{y}) = \mathbf{f}_N(\mathbf{x}, \mathbf{y}), & \forall (\mathbf{x}, \mathbf{y}) \in D \times \Gamma, \\ \boldsymbol{\sigma}_N(\mathbf{x}, \mathbf{y}) = \tilde{E}_N \mathbf{C} \epsilon(\mathbf{u}_N(\mathbf{x}, \mathbf{y})), & \forall (\mathbf{x}, \mathbf{y}) \in D \times \Gamma, \\ \mathbf{u}_N(\mathbf{x}, \mathbf{y}) = 0, & \forall (\mathbf{x}, \mathbf{y}) \in \partial D_0 \times \Gamma, \\ \boldsymbol{\sigma}_N(\mathbf{x}, \mathbf{y}) \mathbf{n} = \mathbf{g}_N(\mathbf{x}, \mathbf{y}), & \forall (\mathbf{x}, \mathbf{y}) \in \partial D_1 \times \Gamma. \end{cases} \quad (3.18)$$

Recall that $\rho : \Gamma \rightarrow R$ is the joint probability density function of random vector Y . We introduce the weighted L^2 -space

$$L_\rho^2(\Gamma) := \{v : \Gamma \rightarrow R \mid \int_\Gamma \rho v^2 d\mathbf{y} < +\infty\}. \quad (3.19)$$

We note that from the norm definition (2.3) it follows

$$\|w\|_{\tilde{m}}^2 = \int_\Gamma \rho(\mathbf{y}) \|w(\cdot, \mathbf{y})\|_m^2 d\mathbf{y} = \|w\|_{L_\rho^2(\Gamma) \otimes H^m(D)}^2, \quad \forall w \in L_\rho^2(\Gamma) \otimes H^m(D). \quad (3.20)$$

It is easy to see that the modified problem (3.16) is equivalent to the following deterministic variational problem: find $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L_\rho^2(\Gamma) \otimes \Sigma_D) \times (L_\rho^2(\Gamma) \otimes V_D)$ such that

$$\begin{cases} a_N(\boldsymbol{\sigma}_N, \boldsymbol{\tau}) - b_N(\boldsymbol{\tau}, \mathbf{u}_N) = 0, & \forall \boldsymbol{\tau} \in L_\rho^2(\Gamma) \otimes \Sigma_D, \\ b_N(\boldsymbol{\sigma}_N, \mathbf{v}) = \ell_N(\mathbf{v}), & \forall \mathbf{v} \in L_\rho^2(\Gamma) \otimes V_D, \end{cases} \quad (3.21)$$

where

$$a_N(\boldsymbol{\sigma}_N, \boldsymbol{\tau}) := \int_\Gamma \rho(\mathbf{y}) \int_D \frac{1}{\tilde{E}_N} \cdot \boldsymbol{\sigma}_N : \mathbf{C}^{-1} \boldsymbol{\tau} d\mathbf{x} d\mathbf{y}, \quad (3.22)$$

$$b_N(\boldsymbol{\tau}, \mathbf{u}_N) := \int_\Gamma \rho(\mathbf{y}) \int_D \boldsymbol{\tau} : \boldsymbol{\epsilon}(\mathbf{u}_N) d\mathbf{x} d\mathbf{y}, \quad (3.23)$$

$$\ell_N(\mathbf{v}) := \int_\Gamma \rho(\mathbf{y}) \int_D \mathbf{f}_N \mathbf{v} d\mathbf{x} d\mathbf{y} + \int_\Gamma \rho(\mathbf{y}) \int_{\partial D_1} \mathbf{g}_N \cdot \mathbf{v} ds d\mathbf{y}. \quad (3.24)$$

The significance of the form (3.21) lies in that it turns the original formulation (2.4) into a deterministic one with perturbations of the Young's modulus \tilde{E} , the body force \mathbf{f} and the surface load \mathbf{g} . Lemma 3.4 shows, if the perturbations or the truncated errors are small enough, we can numerically solve the deterministic problem (3.21) so as to obtain an approximate solution of the original problem (2.4).

Remark 3.3. *In some applications it may be more efficient to numerically solve the problem (3.21) just in a subdomain $\hat{\Gamma} \subset \Gamma$, as, of course, will cause that the corresponding approximation solution has no value in $\Gamma \setminus \hat{\Gamma}$.*

Lemma 3.4. *Suppose that Assumption 3.1 holds and the covariance function, $\text{cov}[\tilde{E}]$, of \tilde{E} is piecewise smooth (cf. Definition 3.1). Then, for sufficiently large N , the modified weak problem (3.16), or its equivalent problem (3.21), admits a unique solution $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L_\rho^2(\Gamma) \otimes \Sigma_D) \times (L_\rho^2(\Gamma) \otimes V_D)$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\tilde{0}} + \|\mathbf{u} - \mathbf{u}_N\|_{\tilde{1}} \lesssim \|\tilde{E} - \tilde{E}_N\|_\infty \cdot \|\boldsymbol{\sigma}\|_{\tilde{0}} + \|\mathbf{f} - \mathbf{f}_N\|_{\tilde{0}} + \|\mathbf{g} - \mathbf{g}_N\|_{\tilde{0}, \partial D_1}, \quad (3.25)$$

where $(\boldsymbol{\sigma}, \mathbf{u}) \in L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D)$ is the solution of the weak problem (2.4).

Moreover, (i) if the covariance functions $\text{cov}[\tilde{E}]$, $\text{cov}[\mathbf{f}]$ and $\text{cov}[\mathbf{g}]$ are piecewise analytic, then there exists a constant $r > 0$, and a constant $C_r > 0$ depending only on $\text{cov}[\tilde{E}]$, $\text{cov}[\mathbf{f}]$, $\text{cov}[\mathbf{g}]$ and r , such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_N|_{\tilde{1}} \lesssim C_r N^{1/2} e^{-rN^{1/2}}. \quad (3.26)$$

(ii) If $\text{cov}[\mathbf{f}]$ and $\text{cov}[\mathbf{g}]$ are piecewise smooth, then for any $s > 0$, there exists $C_s > 0$ depending only on $\text{cov}[\tilde{E}]$, $\text{cov}[\mathbf{f}]$, $\text{cov}[\mathbf{g}]$ and s , such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_N|_{\tilde{1}} \lesssim C_s N^{-s}. \quad (3.27)$$

Proof. We first show the modified problem (3.16) is well-posed. Since the uniform stability conditions for the bilinear form $b(\cdot, \cdot)$ and the linear form $\ell_N(\cdot)$ hold, it suffices to show that \tilde{E}_N is, for sufficiently large N , uniformly bounded with lower bound away from zero a.e. in $D \times \Omega$. In view of Lemma 3.3 and the assumption (1.2), there exists a positive integer N_0 such that, for any $N > N_0$, it holds

$$e'_{\min} \leq \tilde{E}_N \leq e'_{\max} \quad \text{a.e. in } D \times \Omega, \quad (3.28)$$

where e'_{\min} and e'_{\max} are two positive constants depending only on the bounds of \tilde{E} , i.e. e_{\min} and e_{\max} in (1.2). Thus, the corresponding uniform stability conditions of the bilinear form $a_N(\cdot, \cdot)$ follow from those of $a(\cdot, \cdot)$. As a result, the weak problem (3.16) admits a unique solution $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in L_P^2(\Omega; \Sigma_D) \times L_P^2(\Omega; V_D)$ with the stability result

$$\|\boldsymbol{\sigma}_N\|_{\tilde{0}} + |\mathbf{u}_N|_{\tilde{1}} \lesssim \|\mathbf{f}_N\|_{\tilde{0}} + \|\mathbf{g}_N\|_{\tilde{0}, \partial D_1} \quad (3.29)$$

for $N > N_0$.

Next we turn to derive the estimate (3.25). Subtracting the corresponding equations in (2.4) and (3.16), we have

$$\begin{cases} a_N(\boldsymbol{\sigma} - \boldsymbol{\sigma}_N, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \mathbf{u} - \mathbf{u}_N) = a_N(\boldsymbol{\sigma}, \boldsymbol{\tau}) - a(\boldsymbol{\sigma}, \boldsymbol{\tau}), & \forall \boldsymbol{\tau} \in L_P^2(\Omega; \Sigma_D), \\ b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_N, \mathbf{v}) = \ell(\mathbf{v}) - \ell_N(\mathbf{v}), & \forall \mathbf{v} \in L_P^2(\Omega; V_D). \end{cases} \quad (3.30)$$

Then the desired estimate (3.25) follows from the corresponding stability conditions.

By Lemmas 3.2-3.3 and the estimate (3.25), we immediately obtain the estimates (3.26)-(3.27). \square

4 Stochastic hybrid stress finite element methods

In this section, we shall consider two types of stochastic finite element methods for the truncated deterministic variational problem (3.21): $k \times h$ version and $p \times h$ version. We use the PS hybrid stress quadrilateral finite element [17] to discretize the space field and k -version/ p -version finite elements to discretize the stochastic field.

For convenience we assume that the spacial field D is a convex polygon and the stochastic field $\Gamma = \prod_{n=1}^N \Gamma_n$ is bounded (cf. Assumption 3.1).

4.1 Hybrid stress finite element spaces on the spatial field

Let \mathcal{T}_h be a partition of \bar{D} by conventional quadrilaterals with the mesh size $h := \max_{T \in \mathcal{T}_h} h_T$, where h_T is the diameter of quadrilateral $T \in \mathcal{T}_h$. Let $A_i(x_1^{(i)}, x_2^{(i)})$, $1 \leq i \leq 4$, be the four vertices of T , and T_i the sub-triangle of T with vertices A_{i-1}, A_i, A_{i+1} (the index of A_i is modulo 4). We assume that the partition \mathcal{T}_h satisfies the following "shape-regularity" hypothesis : there exist a constant $\zeta > 2$ independent of h such that, for all $T \in \mathcal{T}_h$, it holds

$$h_T \leq \zeta \rho_T, \quad (4.1)$$

where $\rho_T := \min_{1 \leq i \leq 4} \{\text{diameter of circle inscribed in } T_i\}$.

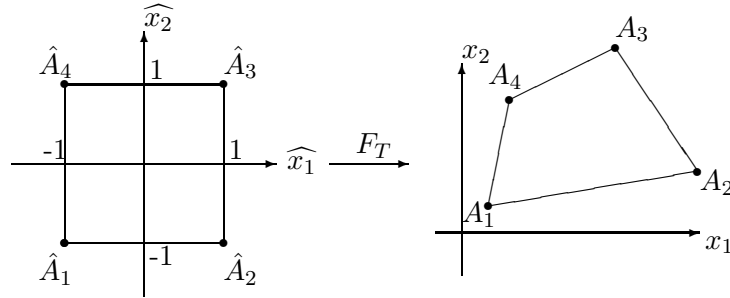


Figure 1: The mapping F_T

Let $\hat{T} = [-1, 1] \times [-1, 1]$ be the reference square with vertices \hat{A}_i , $1 \leq i \leq 4$ (Fig.1). Then exists a unique invertible mapping F_T that maps \hat{T} onto T with $F_T(\hat{A}_i) = A_i$, $1 \leq i \leq 4$. The isoparametric bilinear mapping $(x_1, x_2) = F_T(\hat{x}_1, \hat{x}_2)$ is given by

$$x_1 = a_0 + a_1 \hat{x}_1 + a_2 \hat{x}_1 \hat{x}_2 + a_3 \hat{x}_2, \quad x_2 = b_0 + b_1 \hat{x}_1 + b_2 \hat{x}_1 \hat{x}_2 + b_3 \hat{x}_2, \quad (4.2)$$

where $\hat{x}_1, \hat{x}_2 \in [-1, 1]$ are the local isoparametric coordinates, and

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} := \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ x_1^{(3)} & x_2^{(3)} \\ x_1^{(4)} & x_2^{(4)} \end{pmatrix}.$$

In Pian-Sumiharas hybrid stress finite element (abbr. PS element) method for deterministic plane elasticity problems, the piecewise isoparametric bilinear interpolation is used for the displacement approximation, namely the displacement approximation space $V_{Dh} \subset V_D$ is chosen as

$$V_{Dh} := \{\mathbf{v} \in V_D : \hat{\mathbf{v}} = v|_T \circ F_T \in \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2\}^2, \quad \forall T \in \mathcal{T}_h\}. \quad (4.3)$$

In other words, for $\mathbf{v} = (v, \omega)^T \in V_h$ with nodal values $\mathbf{v}(A_i) = (v_i, \omega_i)^T$ on T , $\hat{\mathbf{v}}$ is of the form

$$\hat{\mathbf{v}} = \begin{pmatrix} V_0 + V_1\hat{x}_1 + V_2\hat{x}_1\hat{x}_2 + V_3\hat{x}_2 \\ W_0 + W_1\hat{x}_1 + W_2\hat{x}_1\hat{x}_2 + W_3\hat{x}_2 \end{pmatrix},$$

where

$$\begin{pmatrix} V_0 & W_0 \\ V_1 & W_1 \\ V_2 & W_2 \\ V_3 & W_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 & \omega_1 \\ v_2 & \omega_2 \\ v_3 & \omega_3 \\ v_4 & \omega_4 \end{pmatrix}.$$

To describe the stress approximation of PS element, we abbreviate the symmetric tensor $\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$ to $\boldsymbol{\tau} = (\tau_{11}, \tau_{22}, \tau_{12})^T$. The 5-parameter stress mode of PS element takes the following form on \hat{T} :

$$\hat{\boldsymbol{\tau}} = \begin{pmatrix} \hat{\tau}_{11} \\ \hat{\tau}_{22} \\ \hat{\tau}_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \hat{x}_2 & \frac{a_2^2}{b_3^2}\hat{x}_1 \\ 0 & 1 & 0 & \frac{b_1^2}{a_1^2}\hat{x}_2 & \hat{x}_1 \\ 0 & 0 & 1 & \frac{b_1}{a_1}\hat{x}_2 & \frac{a_3}{b_3}\hat{x}_1 \end{pmatrix} \boldsymbol{\beta}^\tau \quad \text{for } \boldsymbol{\beta}^\tau = (\beta_1^\tau, \dots, \beta_5^\tau)^T \in R^5. \quad (4.4)$$

Then the corresponding stress approximation space for the PS finite element is

$$\Sigma_{Dh} := \{\boldsymbol{\tau} \in \Sigma_D : \hat{\boldsymbol{\tau}} = \boldsymbol{\tau}|_T \circ F_T \text{ is of form (4.4), } \forall T \in \mathcal{T}_h\}. \quad (4.5)$$

4.2 Stochastic hybrid stress finite element method: $k \times h$ -version

This subsection is devoted to the stability and a priori error analysis for the $k \times h$ -version stochastic hybrid stress finite element method ($k \times h$ -SHSFEM).

4.2.1 $k \times h$ -SHSFEM scheme

We first use the same notations as in [1] to introduce a k -version tensor product finite element space on the stochastic field $\Gamma = \prod_{n=1}^N \Gamma_n \subset R^N$.

Consider a partition of Γ consisting of a finite number of disjoint R^N -boxes, $\gamma = \prod_{n=1}^N (a_n^\gamma, b_n^\gamma)$ with $(a_n^\gamma, b_n^\gamma) \subset \Gamma_n$ and the mesh parameter $k_n := \max_\gamma |b_n^\gamma - a_n^\gamma|$ for $n = 1, 2, \dots, N$.

Let $\mathbf{q} = (q_1, q_2, \dots, q_N)$ be a nonnegative integer multi-index. We define the k -version tensor product finite element space $Y_{\mathbf{k}}^{\mathbf{q}}$ as

$$Y_{\mathbf{k}}^{\mathbf{q}} := \otimes_{n=1}^N Y_{k_n}^{q_n}, \quad Y_{k_n}^{q_n} := \left\{ \varphi : \Gamma_n \rightarrow R : \varphi|_{(a_n^\gamma, b_n^\gamma)} \in \text{span}\{y_n^\alpha : \alpha = 0, 1, \dots, q_n\}, \forall \gamma \right\}. \quad (4.6)$$

The $k \times h$ -SHSFEM scheme for the original weak problem (2.4), or the modified weak problem (3.21), reads as: find $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh}) \in (Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}) \times (Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh})$ such that

$$\begin{cases} a_N(\boldsymbol{\sigma}_{kh}, \boldsymbol{\tau}_{kh}) - b_N(\boldsymbol{\tau}_{kh}, \mathbf{u}_{kh}) = 0, & \forall \boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}, \\ b_N(\boldsymbol{\sigma}_{kh}, \mathbf{v}_{kh}) = \ell_N(\mathbf{v}_{kh}), & \forall \mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}. \end{cases} \quad (4.7)$$

Here we recall that

$$Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh} = \text{span}\{\varphi(\mathbf{y})\boldsymbol{\tau}(\mathbf{x}) : \varphi \in Y_{\mathbf{k}}^{\mathbf{q}}, \boldsymbol{\tau} \in \Sigma_{Dh}\},$$

$$Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh} = \text{span}\{\varphi(\mathbf{y})\mathbf{v}(\mathbf{x}) : \varphi \in Y_{\mathbf{k}}^{\mathbf{q}}, \mathbf{v} \in V_{Dh}\},$$

and V_{Dh}, Σ_{Dh} are defined in (4.3), (4.5), respectively.

4.2.2 Stability

To show the $k \times h$ -SHSFEM scheme (4.7) admits a unique solution, we need some stability conditions. We note that the continuity of $a_N(\cdot, \cdot)$, $b_N(\cdot, \cdot)$ and $\ell_N(\cdot)$ follows from their definitions. Then, according to the theory of mixed methods [3], it suffices to prove the following two discrete versions of the stability conditions.

(**A_h**) Discrete Kernel-coercivity : for any $\boldsymbol{\tau}_{kh} \in Z_{kh}^0 := \{\boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh} : b_N(\boldsymbol{\tau}_{kh}, \mathbf{v}_{kh}) = 0, \forall \mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}\}$, it holds:

$$\|\boldsymbol{\tau}_{kh}\|_0^2 \lesssim a_N(\boldsymbol{\tau}_{kh}, \boldsymbol{\tau}_{kh}). \quad (4.8)$$

(**B_h**) Discrete inf-sup condition : for any $\mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}$, it holds

$$|\mathbf{v}_{kh}|_{\tilde{1}} \lesssim \sup_{0 \neq \boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}} \frac{b_N(\boldsymbol{\tau}_{kh}, \mathbf{v}_{kh})}{\|\boldsymbol{\tau}_{kh}\|_0}. \quad (4.9)$$

To prove the stability condition (\mathbf{A}_h) , we need the following lemma [33]:

Lemma 4.1. *Assume that for any piecewise constant function w , i.e. $w \in L^2(D)$ with $w|_T = \text{const}$, $\forall T \in \mathcal{T}_h$, there exists $\mathbf{v} \in V_{Dh}$ with*

$$\|w\|_0^2 \lesssim \int_D w \operatorname{div} \mathbf{v} \, d\mathbf{x}, \quad \|\mathbf{v}\|_1^2 \lesssim \|w\|_0^2.$$

Then, for any $\boldsymbol{\tau}_h \in \{\boldsymbol{\tau}_h \in \Sigma_{Dh} : \int_D \boldsymbol{\tau}_h : \boldsymbol{\epsilon}(\mathbf{v}_h) d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in V_{Dh}\}$, it holds

$$\|\boldsymbol{\tau}_h\|_0^2 \lesssim \int_D \frac{1}{\widetilde{E}_N} \boldsymbol{\tau}_h : \mathbf{C}^{-1} \boldsymbol{\tau}_h d\mathbf{x}.$$

We note that the assumption of this lemma, which was first used in [34] in the analysis of several quadrilateral nonconforming elements for incompressible elasticity, requires that the quadrilateral mesh is stable for the Stokes element Q1-P0. As we know, the only unstable case for Q1-P0 is the checkerboard mode. Thereupon, any quadrilateral mesh subdivision of D which breaks the checkerboard mode is sufficient for the uniform stability (\mathbf{A}_h) .

Lemma 4.2. *Under the same condition as in Lemma 4.1, the uniform discrete kernel-coercivity condition (\mathbf{A}_h) holds.*

Proof. For any $\boldsymbol{\tau}_{kh} \in Z_{kh}^0$, due to the definitions of spaces $Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}$ and $Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}$ we easily have $\boldsymbol{\tau}_{kh}(\cdot, \mathbf{y}') \in \{\boldsymbol{\tau}_h \in \Sigma_{Dh} : \int_D \boldsymbol{\tau}_h : \boldsymbol{\epsilon}(\mathbf{v}_h) d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in V_{Dh}\}$ for any $\mathbf{y}' \in \Gamma$. From Lemma 4.1 it follows

$$\int_D \boldsymbol{\tau}_{kh}(\cdot, \mathbf{y}') : \boldsymbol{\tau}_{kh}(\cdot, \mathbf{y}') d\mathbf{x} \lesssim \int_D \frac{1}{\widetilde{E}_N(\cdot, \mathbf{y}')} \boldsymbol{\tau}_{kh}(\cdot, \mathbf{y}') : \mathbf{C}^{-1} \boldsymbol{\tau}_{kh}(\cdot, \mathbf{y}') d\mathbf{x}, \quad \forall \mathbf{y}' \in \Gamma, \quad (4.10)$$

which immediately implies (\mathbf{A}_h) . \square

To prove the discrete inf-sup condition \mathbf{B}_h we need the following lemma:

Lemma 4.3. *For any $\mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}$, there exists $\boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}$ such that, for any $T \in \mathcal{T}_h$,*

$$\int_{\Gamma} \rho(\mathbf{y}) \int_T \boldsymbol{\tau}_{kh} : \boldsymbol{\epsilon}(\mathbf{v}_{kh}) d\mathbf{x} d\mathbf{y} = \|\boldsymbol{\tau}_{kh}\|_{0,T}^2 \gtrsim \|\boldsymbol{\epsilon}(\mathbf{v}_{kh})\|_{0,T}^2. \quad (4.11)$$

Proof. The desired result is immediate from Lemma 4.4 in [33]. \square

Lemma 4.4. *The uniform discrete inf-sup condition (\mathbf{B}_h) holds.*

Proof. From Lemma 4.3, for any $\mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}$, there exists $\boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}$ such that

$$\begin{aligned} \|\boldsymbol{\tau}_{kh}\|_{\tilde{0}} |\mathbf{v}_{kh}|_{\tilde{1}} &\lesssim \left(\sum_T \int_{\Gamma} \rho(\mathbf{y}) \int_T \boldsymbol{\tau}_{kh} : \boldsymbol{\tau}_{kh} d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{2}} \left(\sum_T \int_{\Gamma} \rho(\mathbf{y}) \int_T \epsilon(\mathbf{v}_{kh}) : \epsilon(\mathbf{v}_{kh}) d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{2}} \\ &\lesssim \sum_T \int_{\Gamma} \rho(\mathbf{y}) \int_T \boldsymbol{\tau}_{kh} : \boldsymbol{\tau}_{kh} d\mathbf{x} d\mathbf{y} \lesssim \int_{\Gamma} \rho(\mathbf{y}) \int_D \boldsymbol{\tau}_{kh} : \epsilon(\mathbf{v}_{kh}) d\mathbf{x} d\mathbf{y}, \end{aligned}$$

where in the first inequality the equivalence of the seminorm $|\epsilon(\cdot)|_{\tilde{0}}$ and the norm $\|\cdot\|_{\tilde{1}}$ on the space $L_P^2(\Omega; V_D)$ is used. Then the uniform discrete inf-sup condition (\mathbf{B}_h) follows from

$$|\mathbf{v}_{kh}|_{\tilde{1}} \lesssim \frac{\int_{\Gamma} \rho(\mathbf{y}) \int_T \boldsymbol{\tau}_{kh} : \epsilon(\mathbf{v}_{kh}) d\mathbf{x} d\mathbf{y}}{\|\boldsymbol{\tau}_{kh}\|_{\tilde{0}}} \leq \sup_{\boldsymbol{\tau}'_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}} \frac{\int_{\Gamma} \rho(\mathbf{y}) \int_T \boldsymbol{\tau}'_{kh} : \epsilon(\mathbf{v}_{kh}) d\mathbf{x} d\mathbf{y}}{\|\boldsymbol{\tau}'_{kh}\|_{\tilde{0}}}.$$

□

In light of Lemma 4.2 and Lemma 4.4, we immediately obtain the following existence and uniqueness of the $k \times h$ -SHSFEM approximation $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh})$:

Theorem 4.1. *Under the same condition as in Lemma 4.1, the discretization problem (4.7) admits a unique solution $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh}) \in (Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}) \times (Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh})$.*

4.2.3 Uniform error estimation

In what follows we shall derive a priori estimates of the errors $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}}$ and $|\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}}$ which are uniform with respect to the Lamé constant $\lambda \in (0, +\infty)$, where $(\boldsymbol{\sigma}, \mathbf{u}) \in (L_P^2(\Omega; \Sigma_D)) \times (L_P^2(\Omega; V_D))$ is the solution of the weak problem (2.4).

Let $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L_{\rho}^2(\Gamma) \otimes \Sigma_D) \times (L_{\rho}^2(\Gamma) \otimes V_D)$ be the solution of truncated weak problem (3.21). By triangle inequality it holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\tilde{0}} + \|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}}, \quad (4.12)$$

$$|\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}} \leq |\mathbf{u} - \mathbf{u}_N|_{\tilde{1}} + |\mathbf{u}_N - \mathbf{u}_{kh}|_{\tilde{1}}, \quad (4.13)$$

where the perturbation errors, $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\tilde{0}}$ and $|\mathbf{u} - \mathbf{u}_N|_{\tilde{1}}$, are estimated by Lemma 3.4. For the finite element approximation error terms $\|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}}$ and $|\mathbf{u}_N - \mathbf{u}_{kh}|_{\tilde{1}}$, from the stability (\mathbf{A}_h) , (\mathbf{B}_h) and the standard theory of mixed finite element methods [3] it follows

$$\|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u}_N - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim \inf_{\boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_{kh}\|_{\tilde{0}} + \inf_{\mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}} |\mathbf{u}_N - \mathbf{v}_{kh}|_{\tilde{1}}. \quad (4.14)$$

To further estimate the righthand-side terms of the above inequality, we need some regularity of the solution $(\boldsymbol{\sigma}_N, \mathbf{u}_N)$. In fact, it is well-known that the following regularity holds:

$$\|\boldsymbol{\sigma}_N(\cdot, \mathbf{y})\|_1 + \|\mathbf{u}_N(\cdot, \mathbf{y})\|_2 \lesssim \|\mathbf{f}_N(\cdot, \mathbf{y})\|_0 + \|\mathbf{g}_N(\cdot, \mathbf{y})\|_{0, \partial D_1}, \quad \forall \mathbf{y} \in \Gamma. \quad (4.15)$$

On the other hand, in view of (3.28) and the truncated K-L expansions (3.13)-(3.15), and by taking derivatives with respect to y_n in (3.18), standard inductive arguments yield

$$\frac{\|\partial_{y_n}^{q_n+1} \boldsymbol{\sigma}_N(\cdot, \mathbf{y})\|_0}{(q_n+1)!} + \frac{\|\partial_{y_n}^{q_n+1} \mathbf{u}_N(\cdot, \mathbf{y})\|_1}{(q_n+1)!} \lesssim (2\gamma_n)^{q_n+1} (\|\mathbf{f}_N(\cdot, \mathbf{y})\|_0 + \|\mathbf{g}_N(\cdot, \mathbf{y})\|_{0, \partial D_1} + 1), \quad \forall \mathbf{y} \in \Gamma, \quad (4.16)$$

where

$$\gamma_n := \max\left\{\frac{1}{e'_{min}} \sqrt{\tilde{\lambda}_n} \|\tilde{b}_n\|_{L^\infty(D)}, \sqrt{\hat{\lambda}_{in}} \|\hat{b}_{in}\|_0 (i=1,2), \sqrt{\bar{\lambda}_{in}} \|\bar{b}_{in}\|_{0, \partial D_1} (i=1,2)\right\}. \quad (4.17)$$

Then, thanks to $Y_{\mathbf{k}}^{\mathbf{q}} = \otimes_{n=1}^N Y_{k_n}^{q_n}$ and the regularity (4.15)-(4.16), standard interpolation estimation yields

$$\begin{aligned} \inf_{\boldsymbol{\tau}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_{kh}\|_{\tilde{0}} &\lesssim h \|\boldsymbol{\sigma}_N\|_{\tilde{1}} + \sum_{n=1}^N \left(\frac{k_n}{2}\right)^{q_n+1} \frac{\|\partial_{y_n}^{q_n+1} \boldsymbol{\sigma}_N\|_{L^2(\Gamma) \otimes \Sigma_D}}{(q_n+1)!} \\ &\lesssim h + \sum_{n=1}^N (k_n \gamma_n)^{q_n+1}, \end{aligned} \quad (4.18)$$

$$\begin{aligned} \inf_{\mathbf{v}_{kh} \in Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh}} \|\mathbf{u}_N - \mathbf{v}_{kh}\|_{\tilde{1}} &\lesssim h \|\mathbf{u}_N\|_{\tilde{2}} + \sum_{n=1}^N \left(\frac{k_n}{2}\right)^{q_n+1} \frac{\|\partial_{y_n}^{q_n+1} \mathbf{u}_N\|_{L^2(\Gamma) \otimes V_D}}{(q_n+1)!} \\ &\lesssim h + \sum_{n=1}^N (k_n \gamma_n)^{q_n+1}. \end{aligned} \quad (4.19)$$

In light of the estimates (4.14) and (4.18)-(4.19), we immediately obtain the following conclusion.

Theorem 4.2. *Let $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L_\rho^2(\Gamma) \otimes \Sigma_D) \times (L_\rho^2(\Gamma) \otimes V_D)$ and $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh}) \in (Y_{\mathbf{k}}^{\mathbf{q}} \otimes \Sigma_{Dh}) \times (Y_{\mathbf{k}}^{\mathbf{q}} \otimes V_{Dh})$ be the solutions of (3.21) and (4.7), respectively. Then, under the same condition as in Lemma 4.1 and for sufficiently large N , it holds*

$$\|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + \|\mathbf{u}_N - \mathbf{u}_{kh}\|_{\tilde{1}} \lesssim h + \sum_{n=1}^N (k_n \gamma_n)^{q_n+1}. \quad (4.20)$$

Remark 4.1. *We notice that the estimate (4.34) is optimal with respect to the mesh parameters h and $\mathbf{k} = (k_1, k_2, \dots, k_N)$, but not optimal with respect to the polynomial degree $\mathbf{q} = (q_1, q_2, \dots, q_N)$ since it requires $k_n \gamma_n < 1$.*

The above theorem, together with Lemma 3.4, implies the following a priori error estimates for the $k \times h$ -SHSFEM approximation $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh})$.

Theorem 4.3. *Let $(\boldsymbol{\sigma}, \mathbf{u}) \in (L_P^2(\Omega; \Sigma_D)) \times (L_P^2(\Omega; V_D))$ and $(\boldsymbol{\sigma}_{kh}, \mathbf{u}_{kh}) \in (Y_{\mathbf{k}}^q \otimes \Sigma_{Dh}) \times (Y_{\mathbf{k}}^q \otimes V_{Dh})$ be the solutions of (2.4) and (4.7), respectively. Then, under the same conditions as in Theorem 4.2, it holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim N^{1/2} e^{-rN^{1/2}} + h + \sum_{n=1}^N (k_n \gamma_n)^{q_n+1} \quad (4.21)$$

for any $r > 0$ if the covariance functions of \tilde{E} , \mathbf{f} and \mathbf{g} are piecewise analytic, and holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim N^{-s} + h + \sum_{n=1}^N (k_n \gamma_n)^{q_n+1} \quad (4.22)$$

for any $s > 0$ if the covariance functions of \tilde{E} , \mathbf{f} and \mathbf{g} are piecewise smooth.

Remark 4.2. Here we recall that " \lesssim " denotes " $\leq C$ " with C a positive constant independent of λ , h , N , \mathbf{k} .

4.3 Stochastic hybrid stress finite element approximation: $p \times h$ version

As shown in Section 4.2 and Remark 4.1, the $k \times h$ -SHSFEM is based on the k partition of the stochastic field Γ and requires the mesh parameter k_n ($n = 1, 2, \dots, N$) to be sufficiently small so as to acquire optimal error estimates.

In this subsection, we shall introduce a $p \times h$ -version stochastic hybrid stress finite element method ($p \times h$ -SHSFEM), which does not require to refine Γ . We will show this method is of exponential rates of convergence with respect to the degrees of the polynomials used for approximation. To this end, we first assume

$$\tilde{E}_N \in C^0(\Gamma, L^\infty(D)), \quad \mathbf{f}_N \in C^0(\Gamma, L^2(D)), \quad \mathbf{g}_N \in C^0(\Gamma, L^2(\partial D_1)). \quad (4.23)$$

Here

$$C^0(\Gamma, B) := \{v : \Gamma \rightarrow B, v \text{ is continuous in } \mathbf{y} \text{ and } \max_{\mathbf{y} \in \Gamma} \|v(\mathbf{y})\|_B < +\infty\} \quad (4.24)$$

for any Banach space, B , of functions defined in D . The above assumptions indicate that the solution, $(\boldsymbol{\sigma}_N, \mathbf{u}_N)$, of the problem (3.21), satisfies

$$\boldsymbol{\sigma}_N \in C^0(\Gamma, \Sigma_D), \quad \mathbf{u}_N \in C^0(\Gamma, V_D).$$

Let $\mathbf{p} := (p_1, p_2, \dots, p_N)$ be a nonnegative integer multi-index. We define the p -version tensor product finite element space $Z^{\mathbf{p}}$ as

$$Z^{\mathbf{p}} := \otimes_{n=1}^N Z_n^{p_n}, \quad Z_n^{p_n} := \{\varphi : \Gamma_n \rightarrow R : \varphi \in \text{span}\{y_n^\alpha : \alpha = 0, 1, \dots, p_n\}\}. \quad (4.25)$$

Then the $p \times h$ -SHSFEM scheme reads as: find $(\boldsymbol{\sigma}_{ph}, \mathbf{u}_{ph}) \in (Z^{\mathbf{p}} \otimes \Sigma_{Dh}) \times (Z^{\mathbf{p}} \otimes V_{Dh})$ such that

$$\begin{cases} a_N(\boldsymbol{\sigma}_{ph}, \boldsymbol{\tau}_{ph}) - b_N(\boldsymbol{\tau}_{ph}, \mathbf{u}_{ph}) = 0, & \forall \boldsymbol{\tau}_{ph} \in Z^{\mathbf{p}} \otimes \Sigma_{Dh}, \\ b_N(\boldsymbol{\sigma}_{ph}, \mathbf{v}_{ph}) = \ell_N(\mathbf{v}_{ph}), & \forall \mathbf{v}_{ph} \in Z^{\mathbf{p}} \otimes V_{Dh}. \end{cases} \quad (4.26)$$

We note that $Z^{\mathbf{p}}$ is a special case of the k -version tensor product finite element space $Y_{\mathbf{k}}^{\mathbf{q}}$, then, in this sense, the $p \times h$ -SHSFEM can be viewed as a special case of the $k \times h$ -SHSFEM. As a result, the corresponding stability conditions and the existence and uniqueness of the solution of the $p \times h$ -SHSFEM scheme (4.26) follow from those of the $k \times h$ -SHSFEM (cf. Lemma 4.2, Lemma 4.4 and Theorem 4.1).

Following the same routine as in Section 4.2.3 (cf. the estimates (4.12)-(4.14)), we only need to estimate the terms $\inf_{\boldsymbol{\tau}_{ph} \in Z^{\mathbf{p}} \otimes \Sigma_{Dh}} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_{ph}\|_{\tilde{0}}$ and $\inf_{\mathbf{v}_{ph} \in Z^{\mathbf{p}} \otimes V_{Dh}} |\mathbf{u}_N - \mathbf{v}_{ph}|_{\tilde{1}}$. Since

$$\begin{aligned} \inf_{\boldsymbol{\tau}_{ph} \in Z^{\mathbf{p}} \otimes \Sigma_{Dh}} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_{ph}\|_{\tilde{0}} &\lesssim \inf_{\boldsymbol{\tau}_p \in Z^{\mathbf{p}} \otimes \Sigma_D} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_p\|_{\tilde{0}} + \inf_{\boldsymbol{\tau}_h \in L^2_\rho(\Gamma) \otimes \Sigma_{Dh}} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_h\|_{\tilde{0}} \\ &\lesssim \inf_{\boldsymbol{\tau}_p \in Z^{\mathbf{p}} \otimes \Sigma_D} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_p\|_{\tilde{0}} + h \|\boldsymbol{\sigma}_N\|_{\tilde{1}}, \end{aligned} \quad (4.27)$$

$$\begin{aligned} \inf_{\mathbf{v}_{ph} \in Z^{\mathbf{p}} \otimes V_{Dh}} |\mathbf{u}_N - \mathbf{v}_{ph}|_{\tilde{1}} &\lesssim \inf_{\mathbf{v}_p \in Z^{\mathbf{p}} \otimes V_D} |\mathbf{u}_N - \mathbf{v}_p|_{\tilde{1}} + \inf_{\mathbf{v}_h \in L^2_\rho(\Gamma) \otimes V_{Dh}} |\mathbf{u}_N - \mathbf{v}_h|_{\tilde{1}} \\ &\lesssim \inf_{\mathbf{v}_p \in Z^{\mathbf{p}} \otimes V_D} |\mathbf{u}_N - \mathbf{v}_p|_{\tilde{1}} + h \|\mathbf{u}_N\|_{\tilde{2}}, \end{aligned} \quad (4.28)$$

it remains to estimate $\inf_{\boldsymbol{\tau}_p \in Z^{\mathbf{p}} \otimes \Sigma_D} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_p\|_{\tilde{0}}$ and $\inf_{\mathbf{v}_p \in Z^{\mathbf{p}} \otimes V_D} |\mathbf{u}_N - \mathbf{v}_p|_{\tilde{1}}$. Recalling $Z^{\mathbf{p}} = \otimes_{n=1}^N Z_n^{p_n}$, we easily have the following estimates:

$$\inf_{\boldsymbol{\tau}_p \in Z^{\mathbf{p}} \otimes \Sigma_D} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_p\|_{\tilde{0}} \lesssim \sum_{n=1}^N \inf_{\boldsymbol{\tau}_{p_n} \in Z_n^{p_n} \otimes \Sigma_D} \|\boldsymbol{\sigma}_N - \boldsymbol{\tau}_{p_n}\|_{C^0(\Gamma, \Sigma_D)}, \quad (4.29)$$

$$\inf_{\mathbf{v}_p \in Z^{\mathbf{p}} \otimes V_D} |\mathbf{u}_N - \mathbf{v}_p|_{\tilde{1}} \lesssim \sum_{n=1}^N \inf_{\mathbf{v}_{p_n} \in Z_n^{p_n} \otimes V_D} \|\mathbf{u}_N - \mathbf{v}_{p_n}\|_{C^0(\Gamma, V_D)}. \quad (4.30)$$

Then the thing left is to estimate the right hand side terms of the above two inequalities.

Denote $\Gamma_n^* := \prod_{i=1, i \neq n}^N \Gamma_i$, then $\Gamma = \Gamma_n \times \Gamma_n^*$, and for any $\mathbf{y} \in \Gamma$ we denote $\mathbf{y} = (y_n, \mathbf{y}_n^*)$ with $y_n \in \Gamma_n$ and $\mathbf{y}_n^* \in \Gamma_n^*$. We have the following lemma.

Lemma 4.5. *Let $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L^2_\rho(\Gamma) \otimes \Sigma_D) \times (L^2_\rho(\Gamma) \otimes V_D)$ be the solution of the problem (3.21). Then for any $\mathbf{x} \in D$, $\mathbf{y} = (y_n, \mathbf{y}_n^*) \in \Gamma$, the solutions $\boldsymbol{\sigma}_N(x, y_n, \mathbf{y}_n^*)$ and*

$\mathbf{u}_N(x, y_n, y_n^*)$ as functions of y_n , i.e. $\boldsymbol{\sigma}_N : \Gamma_n \rightarrow C^0(\Gamma_n^*; \Sigma_D)$, $\mathbf{u}_N : \Gamma_n \rightarrow C^0(\Gamma_n^*; V_D)$, can be analytically extended to the complex plane

$$\Xi(\Gamma_n; d_n) := \{z \in \mathbb{C}, \text{dist}(z, \Gamma_n) \leq d_n\},$$

with $0 < d_n < \frac{1}{2\gamma_n}$ and γ_n given by (4.17). In addition, for all $z \in \Xi(\Gamma_n; d_n)$, it holds

$$\|\boldsymbol{\sigma}_N(z)\|_{C^0(\Gamma_n^*; \Sigma)} + \|\mathbf{u}_N(z)\|_{C^0(\Gamma_n^*; V_D)} \lesssim \frac{1}{1 - 2d_n\gamma_n} (\|\mathbf{f}_N\|_{C^0(\Gamma; L^2(D))} + \|\mathbf{g}_N\|_{C^0(\Gamma; L^2(\partial D_1))} + 1). \quad (4.31)$$

Proof. Similar to (4.16), for $\mathbf{y} \in \Gamma$, $r \geq 0$ and $n = 1, 2, \dots, N$ it holds

$$\frac{\|\partial_{y_n}^r \boldsymbol{\sigma}_N(\cdot, \mathbf{y})\|_0}{r!} + \frac{|\partial_{y_n}^r \mathbf{u}_N(\cdot, \mathbf{y})|_1}{r!} \lesssim (2\gamma_n)^r (\|\mathbf{f}_N(\cdot, \mathbf{y})\|_0 + \|\mathbf{g}_N(\cdot, \mathbf{y})\|_{0, \partial D_1} + 1). \quad (4.32)$$

For any $y_n \in \Gamma_n$, we define power series

$$\boldsymbol{\sigma}_N(\mathbf{x}, z, y_n^*) = \sum_{r=0}^{\infty} \frac{(z - y_n)^r}{r!} \partial_{y_n}^r \boldsymbol{\sigma}_N(\mathbf{x}, y_n, y_n^*), \quad \mathbf{u}_N(\mathbf{x}, z, y_n^*) = \sum_{r=0}^{\infty} \frac{(z - y_n)^r}{r!} \partial_{y_n}^r \mathbf{u}_N(\mathbf{x}, y_n, y_n^*).$$

then it follows

$$\begin{aligned} \|\boldsymbol{\sigma}_N(\mathbf{x}, z, y_n^*)\|_0 &\leq \sum_{r=0}^{\infty} \frac{|z - y_n|^r}{r!} \|\partial_{y_n}^r \boldsymbol{\sigma}_N(\mathbf{x}, y_n, y_n^*)\|_0, \\ |\mathbf{u}_N(\mathbf{x}, z, y_n^*)|_1 &\leq \sum_{r=0}^{\infty} \frac{|z - y_n|^r}{r!} |\partial_{y_n}^r \mathbf{u}_N(\mathbf{x}, y_n, y_n^*)|_1. \end{aligned}$$

Due to (4.32), we easily know that the above two series converge for all $z \in \Xi(\Gamma_n; d_n)$. Furthermore, by a continuation argument, the functions $\boldsymbol{\sigma}_N$, \mathbf{u}_N can be extended analytically on the whole region $\Xi(\Gamma_n; d_n)$, and the estimate (4.31) follows. \square

In order to estimate the right-hand-side terms of (4.29)(4.30), we need one more lemma by Babuška et al [2].

Lemma 4.6. *Let B be a Banach space, and $L \subset \mathbb{C}$ be a bounded set. Given a function $v \in C^0(L; B)$ which admits an analytic extension in the region of the complex plane $\Xi(L; d) = \{z \in \mathbb{C}, \text{dist}(z, L) \leq d\}$ for some $d > 0$, it holds*

$$\min_{w \in P_p(L) \otimes B} \|v - w\|_{C^0(L; B)} \leq \frac{2}{\varrho - 1} \varrho^{-p} \max_{z \in \Xi(L; d)} \|v(z)\|_B, \quad (4.33)$$

where $P_p(L) := \text{span}(y^s, s = 0, 1, \dots, p)$, $1 < \varrho := \frac{2d}{|L|} + \sqrt{1 + \frac{4d^2}{|L|^2}}$.

In light of (4.27)-(4.30) and Lemmas 4.5-4.6, we immediately obtain the following result.

Theorem 4.4. *Let $(\boldsymbol{\sigma}_N, \mathbf{u}_N) \in (L^2_\rho(\Gamma) \otimes \Sigma_D) \times (L^2_\rho(\Gamma) \otimes V_D)$ and $(\boldsymbol{\sigma}_{ph}, \mathbf{u}_{ph}) \in (Z^p \otimes \Sigma_{Dh}) \times (Z^p \otimes V_{Dh})$ be the solutions of (3.21) and (4.26), respectively. Then, under the same condition as in Lemma 4.1 and for sufficiently large N , it holds*

$$\|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u}_N - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim h + \sum_{n=1}^N \varrho_n^{-p_n}, \quad (4.34)$$

where $\varrho_n = \frac{2d_n}{|\Gamma_n|} + \sqrt{1 + \frac{4d_n^2}{|\Gamma_n|^2}}$ and $0 < d_n < \frac{1}{2\gamma_n}$.

The above theorem, together with Lemma 3.4, implies the following a priori error estimates for the $p \times h$ -SHSFEM approximation $(\boldsymbol{\sigma}_{ph}, \mathbf{u}_{ph})$.

Theorem 4.5. *Let $(\boldsymbol{\sigma}, \mathbf{u}) \in (L^2_P(\Omega; \Sigma_D)) \times (L^2_P(\Omega; V_D))$ and $(\boldsymbol{\sigma}_{ph}, \mathbf{u}_{ph}) \in (Z^p \otimes \Sigma_{Dh}, Z^p \otimes V_{Dh})$ be the solutions of (2.4) and (4.26), respectively. Then, under the same conditions as in Theorem 4.4, it holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim N^{1/2} e^{-rN^{1/2}} + h + \sum_{n=1}^N \varrho_n^{-p_n} \quad (4.35)$$

for any $r > 0$ if the covariance functions of \tilde{E} , \mathbf{f} and \mathbf{g} are piecewise analytic, and holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{kh}\|_{\tilde{0}} + |\mathbf{u} - \mathbf{u}_{kh}|_{\tilde{1}} \lesssim N^{-s} + h + \sum_{n=1}^N \varrho_n^{-p_n} \quad (4.36)$$

for any $s > 0$ if the covariance functions of \tilde{E} , \mathbf{f} and \mathbf{g} are piecewise smooth.

Remark 4.3. *This theorem shows the $p \times h$ -SHSFEM yields exponential rates of convergence with respect to the degrees (p_1, p_2, \dots, p_N) of the polynomials used for approximation.*

5 Numerical examples

In this section we compute two numerical examples to test the performance of the proposed $p \times h$ -version of stochastic hybrid stress finite element method. We note that the $p \times h$ -SHSFEM can be viewed as a particular case of the $k \times h$ version. For convenience we denote

$$e_u := \frac{|\mathbf{u} - \mathbf{u}_h|_{\tilde{1}}}{|\mathbf{u}|_{\tilde{1}}}, \quad e_\sigma := \frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\tilde{0}}}{\|\boldsymbol{\sigma}\|_{\tilde{0}}},$$

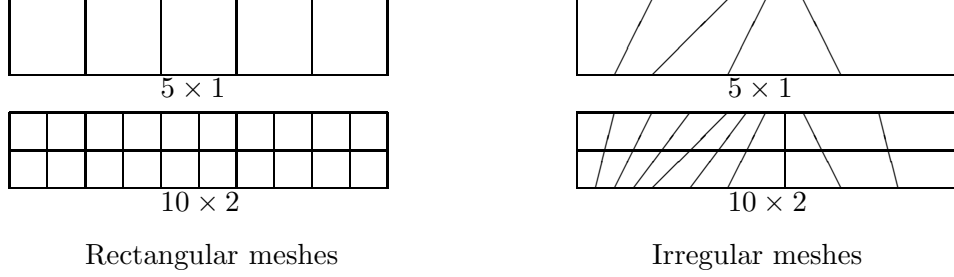


Figure 2: Finite element meshes

where $(\mathbf{u}_h, \boldsymbol{\sigma}_h)$ is the corresponding stochastic finite element approximation to the exact solution $(\mathbf{u}, \boldsymbol{\sigma})$.

Example 1 : stochastic plane stress problem

Set the spatial domain $D = (0, 10) \times (-1, 1)$ with meshes as in Figure 2. The body force \mathbf{f} and the surface traction \mathbf{g} on $\partial D_1 = \{(x_1, x_2) \in [0, 10] \times [-1, 1] : x_1 = 10 \text{ or } x_2 = \pm 1\}$ are given by

$$\mathbf{f} = (0, 0)^T, \quad \mathbf{g}|_{x_1=10} = (-2\tilde{E}x_2, 0)^T, \quad \mathbf{g}|_{x_2=\pm 1} = (0, 0)^T.$$

The exact solution $(\mathbf{u}, \boldsymbol{\sigma})$ is of the form

$$\mathbf{u} = \begin{pmatrix} -2x_1x_2 \\ x_1^2 + \nu(x_2^2 - 1) \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} -2\tilde{E}x_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where \tilde{E} is a uniform random variable on $[500, 1500]$, and we set $\nu = 0.25$.

In the computation we use the exact form of the stochastic coefficient \tilde{E} and take $N = 1$, so there is no truncation error caused by the K-L expansion in the approximation. Numerical results at different meshes and different values of p are listed in Tables 1-2. For comparison we also list results computed by a stochastic finite element called $PC \times h$ method, where the polynomial chaos (PC) method [9] and the PS element method are used in the stochastic field Γ and the space domain D , respectively. In the $PC \times h$ method, p denotes the degree of polynomial chaos. We note that the computational costs of the $PC \times h$ method and the $p \times h$ -SHSFEM are almost the same with the same p .

From the numerical results we can see that the solutions are more accurate with the increasing of p and the refinement of meshes. Especially, $p = 1$ and $p = 2$ for the $p \times h$ -SHSFEM give almost the same results, which implies that the solutions are accurate

enough with respect to the p -version approximation of the stochastic field for given spatial meshes; In these cases, the $p \times h$ -SHSFEM is of first order accuracy in the mesh size h for the displacement approximation and yields quite accurate results for the stress approximation. What's more, we can see that the $p \times h$ -SHSFEM is more accurate than the $PC \times h$ method at the same p .

Table 1: Results for two methods under rectangular meshes: Example 1

Methods	p	e_u				e_σ			
		5×1	10×2	20×4	40×8	5×1	10×2	20×4	40×8
$PC \times h$	4	0.0733	0.0375	0.0204	0.0130	0.0202	0.0202	0.0202	0.0202
	6	0.0728	0.0365	0.0186	0.0098	0.0079	0.0079	0.0079	0.0079
	8	0.0727	0.0364	0.0182	0.0092	0.0033	0.0033	0.0033	0.0033
$p \times h$	0	0.1223	0.1050	0.1003	0.0990	0.2774	0.2774	0.2774	0.2774
	1	0.0727	0.0363	0.0182	0.0091	0	0	0	0
	2	0.0727	0.0363	0.0182	0.0091	0	0	0	0

Table 2: Results for two methods under irregular meshes: Example 1

Methods	p	e_u				e_σ			
		5×1	10×2	20×4	40×8	5×1	10×2	20×4	40×8
$PC \times h$	4	0.1431	0.0637	0.0325	0.0181	0.2632	0.0579	0.0231	0.0203
	6	0.1429	0.0631	0.0314	0.0160	0.2626	0.0549	0.0137	0.0083
	8	0.1429	0.0630	0.0312	0.0156	0.2625	0.0544	0.0117	0.0041
$p \times h$	0	0.1435	0.1160	0.1037	0.0999	0.3684	0.2816	0.2775	0.2774
	1	0.1429	0.0630	0.0311	0.0155	0.2524	0.0509	0.0104	0.0023
	2	0.1429	0.0630	0.0311	0.0155	0.2524	0.0509	0.0104	0.0023

Example 2 : stochastic plane strain problem

The domain Ω and meshes are the same as in Figure 2. The body force $\mathbf{f} = (0, 0)^T$. The surface traction \mathbf{g} on $\partial D_1 = \{(x_1, x_2) \in [0, 10] \times [-1, 1] : x_1 = 10 \text{ or } x_2 = \pm 1\}$ is given by $\mathbf{g}|_{x_1=10} = (-2\tilde{E}x_2, 0)^T$, $\mathbf{g}|_{x_2=\pm 1} = (0, 0)^T$, and the exact solution $(\mathbf{u}, \boldsymbol{\sigma})$ is of the form

$$\mathbf{u} = \begin{pmatrix} -2(1 - \nu^2)x_1x_2 \\ (1 - \nu^2)x_1^2 + \nu(1 + \nu)(x_2^2 - 1) \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} -2\tilde{E}x_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{E} = 1 + \xi^2$, ξ is a standard normal Gaussian random variable.

Similar to Example 1, in the computation we use the exact form of the stochastic coefficient \tilde{E} and take $N = 1$. Numerical results at different meshes, different values of p and different values of Poisson ratio ν are listed in Tables 3-8. For comparison we also

list results computed by a stochastic finite element called $p \times$ bilinear method, where the p -version method and the bilinear element are used in the stochastic field Γ and the space domain D , respectively. We note that the computational costs of the $p \times$ bilinear method and the $p \times h$ -SHSFEM are almost the same.

Tables 3-4 show that the $p \times$ bilinear method deteriorates as $\nu \rightarrow 0.5$ or $\lambda \rightarrow +\infty$, while Tables 5-8 show that the $p \times h$ -SHSFEM yields uniformly accurate results for the displacement and stress approximations. Moreover, $p = 0$ and $p = 2$ give almost the same results, which implies that the solutions are accurate enough with respect to the p -version approximation of the stochastic field for given spatial meshes.

Table 3: Results of e_u for Example 2: $p \times$ bilinear method, $p = 0$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0.5384	0.3061	0.1625	0.0883	0.6854	0.4501	0.2532	0.1356
0.49	0.8516	0.6523	0.4034	0.2175	0.8782	0.7424	0.5218	0.3038
0.499	0.9533	0.9070	0.7856	0.5579	0.9511	0.9145	0.8322	0.6617
0.4999	0.9661	0.9556	0.9365	0.8760	0.9641	0.9550	0.9378	0.8925

Table 4: Results of e_u for Example 2: $p \times$ bilinear method, $p = 2$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0.5384	0.3061	0.1625	0.0883	0.6854	0.4501	0.2532	0.1356
0.49	0.8516	0.6523	0.4034	0.2175	0.9511	0.9145	0.8322	0.6617
0.499	0.9533	0.9070	0.7856	0.5579	0.9511	0.9145	0.8322	0.6617
0.4999	0.9661	0.9556	0.9365	0.8760	0.9641	0.9550	0.9378	0.8925

Table 5: Results of e_u for Example 2: $p \times h$ SHSFEM, $p = 0$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0.0372	0.0186	0.0093	0.0046	0.0676	0.0323	0.0158	0.0079
0.49	0.0488	0.0244	0.0122	0.0061	0.0763	0.0371	0.0183	0.0091
0.499	0.0497	0.0248	0.0124	0.0062	0.0770	0.0375	0.0185	0.0092
0.4999	0.0497	0.0249	0.0124	0.0062	0.0770	0.0375	0.0185	0.0092

Table 6: Results of e_σ for Example 2: $p \times h$ SHSFEM, $p = 0$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0	0	0	0	0.1513	0.0866	0.0450	0.0227
0.49	0	0	0	0	0.1559	0.0877	0.0451	0.0227
0.499	0	0	0	0	0.1563	0.0878	0.0452	0.0227
0.4999	0	0	0	0	0.1564	0.0878	0.0452	0.0227

Table 7: Results of e_u for Example 2: $p \times h$ SHSFEM, $p = 2$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0.0372	0.0186	0.0093	0.0046	0.0676	0.0323	0.0158	0.0079
0.49	0.0488	0.0244	0.0122	0.0061	0.0763	0.0371	0.0183	0.0091
0.49	0.0497	0.0248	0.0124	0.0062	0.0770	0.0375	0.0185	0.0092
0.4999	0.0497	0.0249	0.0124	0.0062	0.0770	0.0375	0.0185	0.0092

Table 8: Results of e_σ for Example 2: $p \times h$ SHSFEM, $p = 2$

ν	Rectangular meshes				Irregular meshes			
	10 \times 2	20 \times 4	40 \times 8	80 \times 16	10 \times 2	20 \times 4	40 \times 8	80 \times 16
0.25	0	0	0	0	0.1513	0.0866	0.0450	0.0227
0.49	0	0	0	0	0.1559	0.0877	0.0451	0.0227
0.499	0	0	0	0	0.0156	0.0878	0.0452	0.0227
0.4999	0	0	0	0	0.1564	0.0878	0.0452	0.0277

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